Math 358 – Foundations of Mathematics **Fall 2008**
9-25-08 Class Notes

Section 2.3
2.3: 53, 55, 72

Indexed Families

As mentioned in the beginning of the set theory topics, what we are covering is technically "naïve" set theory. Normally, sets consist of individual objects, but there is nothing to prevent us from collecting sets themselves together and forming larger sets (whose elements are sets). However, we will resist referring to a set of sets. Instead, in naïve set theory, we will discuss a ***collection*** of sets or a ***family*** of sets. It's just a semantic difference I know, but it does help make it clear that our elements are sets themselves.

In many of these cases (dealing with families of sets) we will not have random sets as elements but rather have a pattern to which sets are elements.

Example 1: Define $\mathscr{A} = \{A_1, A_2, A_3, \ldots\}$ where $A_i = \{n \in \mathbb{N} : n > i\}$ for all $i \in \mathbb{N}$. For example,

$$A_1 = \{n \in \mathbb{N} : n > 1\} = \{2,3,4,\ldots\}$$
$$A_2 = \{n \in \mathbb{N} : n > 2\} = \{3,4,5,\ldots\}$$
$$A_{13} = \{n \in \mathbb{N} : n > 13\} = \{14,15,16,\ldots\}$$

Example 2: Define $\mathscr{B} = \{B_1, B_2, B_3, \ldots\}$ where $B_i = \{x \in \mathbb{R} : i-1 \le x \le i+1\}$ for all $i \in \mathbb{N}$. For example,

$$B_1 = [0,2]$$
$$B_2 = [1,3]$$
$$B_3 = [2,4]$$

Example 3: Define $\mathscr{C} = \{C_1, C_2, C_3, \ldots\}$ where $C_i = \{x \in \mathbb{R} : 0 \le x \le 1/i\}$ for all $i \in \mathbb{N}$. For example,

$$C_1 = [0,1]$$
$$C_2 = [0,1/2]$$
$$C_3 = [0,1/3]$$

Example 4: Define $\mathscr{P} = \{P_1, P_2, P_3, \ldots\}$ where $P_i = \{p \in \mathbb{N} : p \text{ is prime and } p \text{ divides } i\}$ for all $i \in \mathbb{N}$. For example,

$$P_1 = \varnothing$$
$$P_2 = \{2\}$$
$$P_6 = \{2,3\}$$
$$P_{105} = \{3,5,7\}$$

We can take the union and the intersection over a family of sets. This is denoted several different ways. Let $\mathscr{A}$ be a family of sets (say $\mathscr{A} = \{A_1, A_2, A_3, \ldots\}$ or $\mathscr{A} = \{A_i\}$ for short). Then,

$$\bigcup \mathscr{A} = \{x : x \in A_i \text{ for some } A_i \in \mathscr{A}\} = \bigcup_{i=1}^{\infty} A_i \text{, and}$$
$$\bigcap \mathscr{A} = \{x : x \in A_i \text{ for all } A_i \in \mathscr{A}\} = \bigcap_{i=1}^{\infty} A_i .$$

Example 5: See Example 1.

$$\bigcup_{i=1}^{\infty} A_i = \{2,3,4,\ldots\} \text{ and } \bigcap_{i=1}^{\infty} A_i = \varnothing .$$

Example 6: See Example 2.

$$\bigcup_{i=1}^{\infty} B_i = [0,\infty) \text{ and } \bigcap_{i=1}^{\infty} B_i = \varnothing .$$

Example 7: See Example 3.

$$\bigcup_{i=1}^{\infty} C_i = [0,1] \text{ and } \bigcap_{i=1}^{\infty} C_i = \{0\} .$$

Definition: A family of sets is ***pairwise disjoint*** if every pair of distinct sets has empty intersection. In symbols, $\mathscr{A} = \{A_i\}$ is pairwise disjoint if for every $A_i, A_j \in \mathscr{A}$ with $A_i \neq A_j$, $A_i \cap A_j = \varnothing$.

Clearly, if $\mathscr{A} = \{A_i\}$ is pairwise disjoint, then $\bigcap_{i=1}^{\infty} A_i = \varnothing$. But the converse is not true. (E.g. Example 5 above.) In fact, we could even come up with an example of a family of sets $\mathscr{A} = \{A_i\}$ where $\bigcap_{i=1}^{\infty} A_i = \varnothing$ but for each $n \in \mathbb{N}$, $\bigcap_{i=1}^{n} A_i \neq \varnothing$.